

The symmetric signature of a Witt space

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Abstract

Witt spaces are pseudomanifolds for which the middle-perversity intersection homology with rational coefficients is self-dual. We give a new construction of the symmetric signature for Witt spaces which is similar in spirit to the construction given by Miscenko for manifolds. Our construction has all of the expected properties, including invariance under stratified homotopy equivalence.

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Contents

1	Introduction	2
2	Conventions and some background	3
3	The cap product for covering spaces	4
4	Universal Poincaré duality	5
4.1	Lefschetz duality	7
5	The symmetric signature	7
5.1	The symmetric signature for manifolds	7
5.2	Reformulation	9
5.3	Definition of the symmetric signature for F -Witt spaces	11
5.4	Properties of the symmetric signature for F -Witt spaces	13

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6	Technical facts about intersection chains	18
6.1	A colimit formula for intersection chains	18
6.2	Freeness and flatness	19
6.3	Proof of Proposition 5.12	21
6.4	Proof of Proposition 6.3	24

1 Introduction

For a compact oriented m -manifold M (and more generally for a Poincaré duality space) the symmetric signature $\sigma^*(M)$ is an element of the symmetric L -group $L^m(\pi_1(M))$. The symmetric signature was introduced by Miščenko in [26] as a tool for studying the Novikov conjecture, and since then it has become an important part of surgery theory (see [29], for example).

The basic ingredient in the construction of $\sigma^*(M)$ is Poincaré duality on the universal cover. Another situation where Poincaré duality occurs is the middle perversity intersection homology of a certain class of pseudomanifolds, the *Witt spaces* ([33]), so it is natural to ask whether there is a symmetric signature for Witt spaces. The purpose of this paper is to give a positive answer to this question.

There are several other treatments of the symmetric signature for Witt spaces in the literature. Cappell, Shaneson, and Weinberger [6] give a brief description of a construction which uses the work of Quinn and Yamasaki [28, 40]. Further information is given in [36, pages 209–210], but the complete account has not been published. Banagl [4, Section 4] uses the Ph. D. thesis of Thorsten Eppelmann [9] to construct an L-homology fundamental class for a Witt space and then defines the symmetric signature to be the image of this class under the assembly map. However, there are gaps in Eppelmann’s work (Banagl and Laures have informed us that they are working on a corrected version of [9]). Finally, an analytic construction of the symmetric signature (for smoothly stratified Witt spaces) has been given by Albin, Leichtnam, Mazzeo, and Piazza [1, 2].

Our approach has several advantages. It is similar in spirit to that of Miščenko (and thus answers a question in [2]). The actual construction uses only the diagonal map of the pseudomanifold and the cross product on intersection chains, and the supporting results use only the Künneth theorem of [13] and standard facts about intersection chains. We give a simple proof of stratified homotopy invariance; this is proved by a rather intricate analytic argument in [2] and it is not known how to prove it using the approach of [4]. We also give a simple proof of the product formula; to prove this using the approach of [4] one would need to show that Eppelmann’s map $MIP \rightarrow L^\bullet$ is a map of ring spectra up to homotopy.

Applications of the symmetric signature for Witt spaces have been given in [35, 37, 7]. Also, Shmuel Weinberger has pointed out to us that one can use the symmetric signature for Witt spaces to extend [10, Theorem 1.3.2] to Witt spaces.

An argument due to Weinberger (see [2, Proof of Proposition 7.1]) shows that any two definitions of the symmetric signature for Witt spaces must agree rationally if (1) they are bordism invariant and (2) they agree with Miščenko’s definition for smooth manifolds. Thus

all of the known constructions of the symmetric signature agree rationally; it would be interesting to know whether they agree over the integers.

Here is an outline of the paper. In Section 2 we review some background from [18]. For our construction of the symmetric signature we need to know that the intersection homology version of Poincaré duality for the universal cover (which is analogous to what Ranicki [31] calls “universal Poincaré duality”) is given by a cap product; in Section 3 we construct the cap product and in Section 4 we give the proof of universal Poincaré duality. In Section 5 we give the construction of the symmetric signature for Witt spaces and we prove that it has the expected properties. Section 6 gives some technical facts about intersection chains which are needed for the main proofs.

Remark 1.1. We will assume that all pseudomanifolds are oriented (see Section 2 for the definition). In [26, 29] the symmetric signature is defined in the non-orientable case by twisting with the orientation character, but it is not clear how to do that in our situation. For example, the suspension of $\mathbb{R}P^2$ is a non-orientable pseudomanifold which is simply-connected and hence cannot have a non-trivial orientation character $\pi_1(X) \rightarrow \mathbb{Z}_2$.

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2 Conventions and some background

We assume the reader to be conversant with intersection homology theory. Basic textbook introductions to intersection homology include [5, 24, 3], and the original papers [20, 21, 23] are well worth reading. We recommend [17] for an expository introduction to the version of intersection homology considered here and [16] for a more technical account.

Stratified pseudomanifolds and intersection homology. We note here some of our conventions, which sometimes differ from other authors. We continue the conventions of [18] and refer the reader there for more details.

We will work with topological stratified pseudomanifolds X . *Skeleta* of X will be denoted X^i . By a *stratum*, we will mean a connected component of one of the spaces $X^i - X^{i-1}$; a stratum Z is a *singular stratum* if $\dim(Z) < \dim(X)$. X is allowed to have strata of codimension one unless noted otherwise. A *perversity* on X is a function from the set of strata of X to \mathbb{Z} which takes nonsingular strata to 0. This is a much more general definition than that in [20, 21]; on the rare occasions when we want to refer to perversities as defined in [20, 21] we will call them “classical perversities.”

An *orientation* of a stratified pseudomanifold is a choice of orientations for the top strata.

In the literature, there are several non-equivalent definitions of intersection homology with general perversities. We use the version in [17, 16] (which is equivalent to that in [32]). In [17, 16] this version of intersection homology (with F coefficients) was denoted $I^{\bar{p}}H_*(X; F_0)$

but (as in [18]) we will denote it simply by $I^{\bar{p}}H_*(X; F)$. This version of intersection homology agrees with the definition in [20, 21] when \bar{p} is a classical perversity and X has no strata of codimension one.

We let $D\bar{p}$ denote the complementary perversity to \bar{p} , i.e. $D\bar{p}(Z) = \text{codim}(Z) - 2 - \bar{p}(Z)$.

We direct the reader to [18, Section 4] for intersection cochains and for the chain-level versions of intersection (co)homology cup and cap products.

Signs. We include a sign in the Poincaré duality isomorphism (see [15, Section 4.1]). Except for this we follow the signs in [8], which means that we use the Koszul convention everywhere except in the definition of the coboundary on cochains. Dold's convention for the differential of a cochain (see [8, Remark VI.10.28]) is

$$(\delta\alpha)(x) = -(-1)^{|\alpha|}\alpha(\partial x).$$

This convention is necessary in order for the evaluation map to be a chain map.

3 The cap product for covering spaces

Let $p : \tilde{X} \rightarrow X$ be a regular cover with group π . For any subset A of X we write \tilde{A} for $p^{-1}(A)$.

Notation 3.1. 1. Given a perversity \bar{p} on X , the perversity on \tilde{X} which takes a stratum S to $\bar{p}(p(S))$ will also be denoted by \bar{p} .

2. We will write $I_{\bar{p}}\bar{C}^*(\tilde{X}; F)$ for $\text{Hom}_{F[\pi]}(I^{\bar{p}}C_*(\tilde{X}; F), F[\pi])$ and $I_{\bar{p}}\bar{H}^*(\tilde{X}; F)$ for the cohomology groups of this complex.

Remark 3.2. If the covering $p : \tilde{X} \rightarrow X$ is trivial (i.e., if it is isomorphic to the projection $\pi \times X \rightarrow X$) then $I_{\bar{p}}\bar{H}^*(\tilde{X}; F)$ is $\text{Hom}_F(I_{\bar{p}}H_*(X, F), F[\pi])$.

In this section we define a cap product

$$I_{\bar{q}}\bar{H}^i(\tilde{X}; F) \otimes I^{\bar{r}}H_j(X; F) \rightarrow I^{\bar{p}}H_{j-i}(\tilde{X}; F)$$

when $D\bar{r} \geq D\bar{p} + D\bar{q}$ and F is a field.

The construction follows the general outline of [18, Section 4], so we begin by constructing a suitable algebraic diagonal map. For a left $F[\pi]$ -module M , let M^t denote the right $F[\pi]$ -module structure on M induced by the standard involution of $F[\pi]$.

Let

$$\tilde{d} : I^{\bar{r}}H_*(X; F) \rightarrow H_*(I^{\bar{p}}C_*(\tilde{X}; F)^t \otimes_{F[\pi]} I^{\bar{q}}C_*(\tilde{X}; F))$$

be the composition

$$\begin{aligned} I^{\bar{r}}H_*(X; F) &\xleftarrow{\cong} H_*(F \otimes_{F[\pi]} I^{\bar{r}}C_*(\tilde{X}; F)) \\ &\xrightarrow{1 \otimes d} H_*(F \otimes_{F[\pi]} I^{\mathcal{Q}_{\bar{p}, \bar{q}}}C_*(\tilde{X} \times \tilde{X}; F)) \\ &\xleftarrow{\cong} H_*(F \otimes_{F[\pi]} (I^{\bar{p}}C_*(\tilde{X}; F) \otimes_F I^{\bar{q}}C_*(\tilde{X}; F))) \\ &\cong H_*(I^{\bar{p}}C_*(\tilde{X}; F)^t \otimes_{F[\pi]} I^{\bar{q}}C_*(\tilde{X}; F)). \end{aligned}$$

Here d is the diagonal map given by [18, Proposition 4.2.1]. The first isomorphism is given by Proposition 6.1.3 below. The second isomorphism is given by the Künneth theorem [18, Theorem 3.1] and Proposition 6.5 below. The third isomorphism is elementary.

Suppose now that $\alpha \in I_{\bar{q}}\bar{H}^*(\tilde{X}; F)$ and that $x \in I^{\bar{r}}H_*(X; F)$. We note that $H_*(I^{\bar{p}}C_*(\tilde{X}; F)^t)$ is the same F -vector space as $I^{\bar{p}}H_*(\tilde{X}; F)$, and we define $\alpha \frown x \in I^{\bar{p}}H_*(\tilde{X}; F)$ by

$$\alpha \frown x = (1 \otimes \alpha)\tilde{d}(x).$$

Explicitly, if $\tilde{d}(x)$ is represented by a cycle $\sum_a y_a \otimes z_a$, then $\alpha \frown x$ is represented by $(-1)^{|\alpha||y_a|} \sum_a y_a \alpha(z_a) \in I^{\bar{p}}C_*(\tilde{X}; F)^t$.

If π is trivial this construction reduces to the cap product defined in [18, Section 4.3].

Similarly, when A and B are open subsets of X , we can define the relative cap product

$$I_{\bar{q}}\bar{H}^i(\tilde{X}, \tilde{A}; F) \otimes I^{\bar{r}}H_j(X, A \cup B; F) \rightarrow I^{\bar{p}}H_{j-i}(\tilde{X}, \tilde{B}; F).$$

In the next section, we will (implicitly) use the fact that [18, Propositions 4.16 and 4.19] have analogues for the cap product discussed in this section. We leave it to the reader to check that the proofs in [18] go through in this situation. We will also need an analogue of [18, Proposition 4.21], and for this we need to define the cohomology cross product

$$\times : H^*(M; F) \otimes I_{\bar{p}}\bar{H}^*(\tilde{X}; F) \rightarrow I_{\bar{p}}\bar{H}^*(M \times \tilde{X}; F)$$

in the special case where the covering $p : \tilde{X} \rightarrow X$ is trivial; we define it to be the composite

$$\begin{aligned} H^*(M; F) \otimes I_{\bar{p}}\bar{H}^*(\tilde{X}; F) &\cong \text{Hom}_F(H_*(M; F), F) \otimes \text{Hom}_F(I^{\bar{p}}H_*(X; F), F[\pi]) \\ &\rightarrow \text{Hom}_F(H_*(M; F) \otimes I^{\bar{p}}H_*(X; F), F[\pi]) \cong \text{Hom}_F(I^{\bar{p}}H_*(M \times X; F), F[\pi]) \\ &\cong I_{\bar{p}}\bar{H}^*(M \times \tilde{X}; F), \end{aligned}$$

using Remark 3.2.

Remark 3.3. This is an isomorphism when $H_*(M; F)$ is finitely generated.

4 Universal Poincaré duality

In this section, we consider “universal” Poincaré duality—the duality for regular coverings of stratified pseudomanifolds. For manifolds, universal duality plays an important role in surgery theory and in the definition of L -theory invariants, such as the symmetric signature; see [31, Section 4.5] and [29].

Let F be a field, and let X be an F -oriented n -dimensional stratified pseudomanifold, possibly noncompact. Let $p : \tilde{X} \rightarrow X$ be a regular cover with group π . For each compact $K \subset X$, let Γ_K be the fundamental class of $I^0H_n(X, X - K; F)$ (see [18, Definition 5.8]) and let \bar{p}, \bar{q} be complementary perversities, i.e. $\bar{p}(Z) + \bar{q}(Z) = \text{codim}(Z) - 2$ for each singular stratum Z .

Let

$$\mathcal{D} : \varinjlim_K I_{\bar{p}}\bar{H}^i(\tilde{X}, \tilde{X} - \tilde{K}; F) \rightarrow I^{\bar{q}}H_{n-i}(\tilde{X}; F)$$

be the map obtained by passage to the direct limit from

$$(-1)^{in} \cdot \cap \Gamma_K : I_{\bar{p}} \bar{H}^i(\tilde{X}, \tilde{X} - \tilde{K}; F) \rightarrow I^{\bar{q}} H_{n-i}(\tilde{X}; F)$$

(compare the discussion in [18] that comes before the statement of Theorem 6.3, and see [15, Section 4.1] for the sign).

Theorem 4.1 (Universal Poincaré duality). *Let X be an F -oriented stratified pseudomanifold, possibly noncompact and possibly with codimension one strata, let $p : \tilde{X} \rightarrow X$ be a regular π -covering of X , and let \bar{p}, \bar{q} be complementary perversities. Then \mathcal{D} is an isomorphism.*

The proof will occupy the remainder of this section.

The proof follows the same outline as the proof of [18, Theorem 6.2]. First we need the following analogue of [18, Lemma 6.4].

Lemma 4.2. *Let L be a compact $k-1$ dimensional stratified pseudomanifold. If the conclusion of Theorem 4.1 holds for L with the trivial covering map $\pi \times L \rightarrow L$ then it also holds for cL with the trivial covering map $\pi \times cL \rightarrow cL$.*

The proof of this lemma is the same as that of [18, Lemma 6.4], except that Remark 3.2 above should be used in place of [18, Remark 4.9].

Next we need the following analogue of [18, Lemma 6.6].

Lemma 4.3. *Suppose that the conclusion of Theorem 4.1 holds for the compact F -oriented stratified $k-1$ pseudomanifold L with the trivial covering map $\pi \times L \rightarrow L$. Let M be an F -oriented unstratified $n-k$ manifold, and assume that $H_*(M, M-C; F)$ is finitely generated for a cofinal collection of compact subsets C . Give $M \times cL$ the product stratification and the product orientation. Then the conclusion of Theorem 4.1 holds for $M \times cL$ with the trivial covering map $\pi \times M \times cL \rightarrow M \times cL$.*

The proof is the same as that of [18, Lemma 6.6], except that the relative version of Remark 3.3 above should be used in place of the relative version of [18, Remark 4.20].

The next part of the proof of [18, Theorem 6.3] is a Zorn's lemma argument using an induction over depth. The analogous argument works in our situation because of the following observations:

- In order to construct the Mayer-Vietoris sequence for $I_{\bar{p}} \bar{H}^*$ it suffices to know that if $A \subset B$ are open subsets of X then the inclusion $I^{\bar{p}} C_*(\tilde{A}; F) \hookrightarrow I^{\bar{p}} C_*(\tilde{B}; F)$ is split as a map of $F[\pi]$ -modules. This in turn follows from the proof of [12, Proposition 2.9] (use the construction in that proof with X taken to be B and the ordered open cover taken to be (A, B)).
- In the situation where Lemmas 4.2 and 4.3 are needed, $M \times cL$ is contained in a distinguished neighborhood, so in particular the restriction of the covering map $p : \tilde{X} \rightarrow X$ to a cover of $M \times cL$ is trivial.

- Moreover, the M 's that occur in the proof are open subsets of Euclidean space. Such an M is a PL manifold, so we can take C in Lemma 4.3 to be a compact PL subspace and then $H_*(M, M - C; F)$ is finitely generated by Poincaré-Lefschetz duality ([8, Proposition VIII.7.2]).

It remains to consider the analogues of [18, Lemmas 6.8 and 6.9]. All of the steps in the proof of [18, Lemma 6.8] go through without change in our situation. For the analogue of [18, Lemma 6.9], we need to know that the map

$$\begin{aligned} \lambda : H_*(F \otimes_{F[\pi]} \varinjlim_{W \in \mathcal{C}} I^{Q_{\bar{p}, \bar{q}}} C_*(\tilde{W} \times \tilde{W}, \tilde{W} \times (\tilde{W} - \tilde{K} \cup \tilde{L}))) \\ \rightarrow H_*(F \otimes_{F[\pi]} I^{Q_{\bar{p}, \bar{q}}} C_*(\tilde{Y}, \tilde{Y} - (\tilde{X} \times (\tilde{K} \cup \tilde{L})))) \end{aligned}$$

is an isomorphism; this is immediate from Proposition 6.1.2 below.

Remark 4.4. It seems likely that the proof of Poincaré duality given in this section generalizes to local coefficient systems defined on $X - X^{n-1}$ (cf. [21]).

4.1 Lefschetz duality

Lefschetz duality also generalizes to the universal setting, yielding the following corollary to Theorem 4.1. The proof follows from Theorem 4.1 just as [18, Theorem 7.10] follows from [18, Theorem 6.3]. See [18, Section 7.1] for the definition of topological ∂ -stratified pseudomanifold.

Theorem 4.5 (Universal Lefschetz Duality). *Let X be an n -dimensional compact ∂ -stratified pseudomanifold such that $X - \partial X$ is F -oriented. Let $p : \tilde{X} \rightarrow X$ be a regular π -covering, and let \bar{p}, \bar{q} be complementary perversities. Then the cap product with Γ_X is an isomorphism $I_{\bar{p}} \bar{H}^i(\tilde{X}; F) \rightarrow I^{\bar{q}} H_{n-i}(\tilde{X}; F)$.*

5 The symmetric signature

In Section 5.1, we review the construction of the symmetric signature for compact oriented manifolds. Section 5.2 gives a reformulation that is convenient for our purposes. In Section 5.3, we construct the symmetric signature for F -Witt spaces, and in Section 5.4 we show that it has the expected properties.

5.1 The symmetric signature for manifolds

Given a closed oriented manifold M of dimension m , a discrete group π , and a map $f : M \rightarrow B\pi$, the symmetric signature $\sigma^*(f)$ is an element of the symmetric L -group $L^m(\mathbb{Z}[\pi])$. We begin by recalling the definition of this group from [29, Section 1] (with a variation introduced in [38]), which requires some preliminary definitions.

Let R be a ring with involution and let C be a chain complex of left R -modules. The involution gives a chain complex C^t of right R -modules. There is a chain map, called the *slant product*

$$\backslash : \text{Hom}_R(C, R) \otimes_{\mathbb{Z}} (C^t \otimes_R C) \rightarrow C^t$$

defined by $\alpha \otimes x \otimes y \rightarrow (-1)^{|\alpha||x|} x \alpha(y)$ (cf. [8, Section VII.11]).

Definition 5.1. A chain complex C over R is *finite* if it is free and finitely generated over R in each degree and nonzero only in finitely many degrees. It is *homotopy finite* if it is chain homotopy equivalent over R to a finite chain complex over R .

Let W be the standard $\mathbb{Z}[\mathbb{Z}/2]$ -free resolution of \mathbb{Z} . Let $\iota \in H_0(W)$ be the generator.

Definition 5.2. An n -dimensional *symmetric Poincaré complex* over R is a pair (C, ϕ) , where C is a homotopy finite chain complex over R and ϕ is a $\mathbb{Z}/2$ -equivariant chain map

$$\phi : W \rightarrow C^t \otimes_R C$$

which raises degrees by n , such that the slant product with $\phi_*(\iota)$ is an isomorphism

$$H^*(\text{Hom}_R(C, R)) \rightarrow H_{n-*}(C^t).$$

(Note that $H_*(C^t) = H_*(C)$ as graded abelian groups.)

The concept of a *symmetric Poincaré pair* (which we will denote by $((D, \Phi), (C, \phi))$) is defined in a similar way ([29, Definition 1.7]).

Definition 5.3. 1. Given symmetric Poincaré complexes (C, ϕ) and (C', ϕ') over R , the *direct sum* $(C, \phi) \oplus (C', \phi')$ is the symmetric Poincaré complex $(C \oplus C', \psi)$, where ψ is the composite

$$W \xrightarrow{\text{diag}} W \oplus W \xrightarrow{\phi \oplus \phi'} (C^t \otimes_R C) \oplus (C'^t \otimes_R C') \hookrightarrow (C \oplus C')^t \otimes_R (C \oplus C').$$

2. (C, ϕ) and (C', ϕ') are *bordant* if there is a symmetric Poincaré pair $((D, \Phi), (C, \phi) \oplus (C', -\phi'))$.
3. $L^n(R)$ is the bordism group of n -dimensional symmetric Poincaré complexes (with addition given by direct sum).

Remark 5.4. The definition of symmetric Poincaré complex in [29, Section 1] requires C to be a finite chain complex over R and not just homotopy finite. It's easy to check (using the proof of [38, Lemma 3.4]) that the L groups in Definition 5.3.3 are the same as those in [29].

Remark 5.5. A $\mathbb{Z}/2$ -equivariant chain map $W \rightarrow C^t \otimes_R C$ that raises degrees by n represents an element of $H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, C^t \otimes_R C))$. If (C, ϕ) is a symmetric Poincaré complex and $\psi : W \rightarrow C^t \otimes_R C$ represents the same homology class as ϕ (i.e., if ψ is $\mathbb{Z}/2$ -equivariantly chain homotopic to ϕ), then (C, ψ) is a symmetric Poincaré complex that is homotopy equivalent to (C, ϕ) ([29, Definition 1.6(ii)]) and therefore represents the same element of $L^n(R)$ (by [29, Proposition 1.13]).

Now let $f : M \rightarrow B\pi$ be a map with M compact oriented of dimension n and π discrete. Let \tilde{M} be the induced cover of M . The singular chain complex $C_*(\tilde{M})$ is homotopy finite over $\mathbb{Z}[\pi]$ (for example, by [39, Corollary 5.3]). Choose a representative $\xi \in S_n(M)$ for the fundamental class of M , and let ϕ_M be the composite

$$\begin{aligned} W \cong W \otimes \mathbb{Z} &\xrightarrow{1 \otimes \xi} W \otimes C_*(M) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} (W \otimes C_*(\tilde{M})) \\ &\xrightarrow{1 \otimes \text{EAW}} \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} (C_*(\tilde{M}) \otimes C_*(\tilde{M})) \cong (C_*(\tilde{M}))^t \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{M}), \end{aligned}$$

where EAW is the extended Alexander-Whitney map (which can be constructed by an acyclic models argument). The symmetric signature $\sigma^*(f)$ is the class in $L^n(\mathbb{Z}[\pi])$ represented by the symmetric Poincaré complex $(C_*(\tilde{M}), \phi_M)$. This is independent of the choice of ξ by Remark 5.5.

5.2 Reformulation

In this section we give an equivalent definition of the symmetric signature that does not use the extended Alexander-Whitney map (see Corollary 5.8). We use the notation of the previous section.

Our first result shows that EAW can be replaced, for our purposes, by the diagram

$$W \otimes C_*(\tilde{M}) \xrightarrow{\varepsilon \otimes 1} C_*(\tilde{M}) \xrightarrow{d} C_*(\tilde{M} \times \tilde{M}) \xleftarrow{\times} C_*(\tilde{M}) \otimes C_*(\tilde{M}),$$

where ε is the augmentation and d is induced by the diagonal map.

Proposition 5.6. *The diagram*

$$\begin{array}{ccc} W \otimes C_*(\tilde{M}) & \xrightarrow{\text{EAW}} & C_*(\tilde{M}) \otimes C_*(\tilde{M}) \\ \varepsilon \otimes 1 \downarrow & & \downarrow \times \\ C_*(\tilde{M}) & \xrightarrow{d} & C_*(\tilde{M} \times \tilde{M}) \end{array}$$

commutes up to $(\mathbb{Z}/2 \times \pi_1 M)$ -equivariant chain homotopy (where $\pi_1 M$ acts diagonally on $C_(\tilde{M}) \otimes C_*(\tilde{M})$ and $C_*(\tilde{M} \times \tilde{M})$).*

We defer the proof to the end of the section.

The map

$$C_*(\tilde{M}) \otimes C_*(\tilde{M}) \xrightarrow{\times} C_*(\tilde{M} \times \tilde{M})$$

is a quasi-isomorphism whose domain and target are free over $\mathbb{Z}[\pi]$, hence it is a chain homotopy equivalence over $\mathbb{Z}[\pi]$ (see, for example, [22, Exercise IV.4.2]), and we obtain a quasi-isomorphism

$$\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} (C_*(\tilde{M}) \otimes C_*(\tilde{M})) \xrightarrow{1 \otimes \times} \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{M} \times \tilde{M}).$$

This in turn induces an isomorphism

$$\begin{aligned} H_*(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, (C_*(\tilde{M}))^t \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{M}))) &\cong H_*(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} (C_*(\tilde{M}) \otimes C_*(\tilde{M})))) \\ &\rightarrow H_*(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{M} \times \tilde{M}))), \end{aligned}$$

which we denote by Υ . Let

$$c_f \in H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{M} \times \tilde{M})))$$

be the class represented by the composite

$$W \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\xi} C_*(M) \cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{M}) \xrightarrow{1 \otimes d} \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{M} \times \tilde{M}).$$

Our next result is an easy consequence of Proposition 5.6.

Proposition 5.7. Υ takes the homology class of ϕ_M to c_f . □

Combining this with Remark 5.5 gives:

Corollary 5.8. *If $\psi : W \rightarrow (C_*(\tilde{M}))^t \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{M})$ is any $\mathbb{Z}/2$ -equivariant chain map whose homology class is $\Upsilon^{-1}(c_f)$ then $(C_*(\tilde{M}), \psi)$ is a representative for $\sigma^*(f)$.*

Proof of Proposition 5.6. We use the formula for EAW given in [25, Definition 2.10(a) and Remark 2.11(a)]. A similar formula gives a natural transformation

$$W \otimes C_*(X \times Y) \rightarrow C_*(X) \otimes C_*(Y)$$

which we denote by EZ. EAW factors as

$$W \otimes C_*(X) \xrightarrow{1 \otimes d} W \otimes C_*(X \times X) \xrightarrow{\mathrm{EZ}} C_*(X) \otimes C_*(X),$$

so to prove the proposition it suffices to show that the diagram

$$\begin{array}{ccc} W \otimes C_*(\tilde{M} \times \tilde{M}) & \xrightarrow{\mathrm{EZ}} & C_*(\tilde{M}) \otimes C_*(\tilde{M}) \\ & \searrow \varepsilon \otimes 1 & \downarrow \times \\ & & C_*(\tilde{M} \times \tilde{M}) \end{array}$$

commutes up to $(\mathbb{Z}/2 \times \pi_1 M)$ -equivariant chain homotopy. Since the map

$$W \otimes C_*(\tilde{M}) \otimes C_*(\tilde{M}) \xrightarrow{1 \otimes \times} W \otimes C_*(\tilde{M} \times \tilde{M})$$

is a $(\mathbb{Z}/2 \times \pi_1 M)$ -equivariant chain homotopy equivalence, it suffices to show that the composites

$$W \otimes C_*(\tilde{M}) \otimes C_*(\tilde{M}) \xrightarrow{1 \otimes \times} W \otimes C_*(\tilde{M} \times \tilde{M}) \xrightarrow{\mathrm{EZ}} C_*(\tilde{M}) \otimes C_*(\tilde{M}) \xrightarrow{\times} C_*(\tilde{M} \times \tilde{M})$$

and

$$W \otimes C_*(\tilde{M}) \otimes C_*(\tilde{M}) \xrightarrow{1 \otimes \times} W \otimes C_*(\tilde{M} \times \tilde{M}) \xrightarrow{\varepsilon \otimes 1} C_*(\tilde{M} \times \tilde{M})$$

are equal. As $(\varepsilon \otimes 1)(1 \otimes \times) = \varepsilon \otimes \times = (1 \otimes \times)(\varepsilon \otimes 1)$, this in turn follows from the fact that the diagram

$$\begin{array}{ccc} W \otimes C_*(X) \otimes C_*(Y) & \xrightarrow{1 \otimes \times} & W \otimes C_*(X \times Y) \\ & \searrow \varepsilon \otimes 1 & \downarrow \text{EZ} \\ & & C_*(X) \otimes C_*(Y) \end{array}$$

commutes (which is easily checked from the definitions of EZ and \times). \square

5.3 Definition of the symmetric signature for F -Witt spaces

In this section we use coefficients in a field F .

We will use the definition of F -Witt space from [14, page 1271]; in particular F -Witt spaces are PL, compact, oriented have no codimension one strata. In fact everything we do would go through without change for topological F -Witt spaces, except for the proofs of Proposition 5.12 (which is probably still true in the topological setting) and Theorem 5.17.1 (see Remark 5.18).

Let X be an F -Witt space, let $f : X \rightarrow B\pi$ be a map, and let \tilde{X} be the induced cover of X . Recall that the upper middle perversity \bar{n} is defined by

$$\bar{n}(Z) = \begin{cases} 0, & \text{if } \text{codim}(Z) \leq 1, \\ \lceil \frac{\text{codim}(Z)-2}{2} \rceil, & \text{if } \text{codim}(Z) \geq 2. \end{cases}$$

In order to define the symmetric signature of f we follow the pattern of the previous section.

By Proposition 6.5, the map

$$F \otimes_{F[\pi]} (I^{\bar{n}}C_*(\tilde{X}; F) \otimes_F I^{\bar{n}}C_*(\tilde{X}; F)) \xrightarrow{1 \otimes \times} F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{n}}}C_*(\tilde{X} \times \tilde{X}; F)$$

is a quasi-isomorphism, which is evidently $\mathbb{Z}/2$ -equivariant. Combining this with the isomorphism

$$(I^{\bar{n}}C_*(\tilde{X}; F))^t \otimes_{F[\pi]} I^{\bar{n}}C_*(\tilde{X}; F) \cong F \otimes_{F[\pi]} (I^{\bar{n}}C_*(\tilde{X}; F) \otimes_F I^{\bar{n}}C_*(\tilde{X}; F)),$$

we obtain an isomorphism

$$\begin{aligned} \Upsilon : H_*(\text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, (I^{\bar{n}}C_*(\tilde{X}; F))^t \otimes_{F[\pi]} I^{\bar{n}}C_*(\tilde{X}; F))) \\ \xrightarrow{\cong} H_*(\text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{n}}}C_*(\tilde{X} \times \tilde{X}; F))). \end{aligned}$$

Next we construct a class

$$c_f \in H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{n}}}C_*(\tilde{X} \times \tilde{X}; F))).$$

Notation 5.9. Let $b_X \in H_n(F \otimes_{F[\pi]} I^{\bar{0}}C_*(\tilde{X}; F))$ map to the fundamental class Γ_X under the isomorphism

$$H_*(F \otimes_{F[\pi]} I^{\bar{0}}C_*(\tilde{X}; F)) \rightarrow I^{\bar{0}}H_*(X; F)$$

given by Proposition 6.1.3.

Let ζ be a cycle representing b_X and let c_f be the class represented by the composite

$$W \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\zeta} F \otimes_{F[\pi]} I^{\bar{0}}C_*(\tilde{X}; F) \xrightarrow{1 \otimes d} F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{n}}}C_*(\tilde{X} \times \tilde{X}; F).$$

Proposition 5.10. *Let*

$$\psi : W \rightarrow (I^{\bar{n}}C_*(\tilde{X}; F))^t \otimes_{F[\pi]} I^{\bar{n}}C_*(\tilde{X}; F)$$

be a $\mathbb{Z}/2$ -equivariant chain map that represents $\Upsilon^{-1}(c_f)$. Then

$$(I^{\bar{n}}C_*(\tilde{X}; F), \psi)$$

is a symmetric Poincaré complex.

Before proving this we give

Definition 5.11. The *symmetric signature* of $f : X \rightarrow B\pi$, denoted $\sigma_{\text{Witt}}^*(f)$, is the class in $L^n(F[\pi])$ represented by $(I^{\bar{n}}C_*(\tilde{X}; F), \psi)$, with ψ as in Proposition 5.10.

The first step in proving Proposition 5.10 is the following result, which will be proved in Section 6.3.

Proposition 5.12. *Let X be a compact PL ∂ -stratified pseudomanifold. Let \tilde{X} be a regular covering of X with group π . For any perversity \bar{p} , the chain complex $I^{\bar{p}}C_*(\tilde{X}; F)$ is homotopy finite over $F[\pi]$.*

According to Definition 5.2, to complete the proof of Proposition 5.10 we need to show that the slant product with $\psi_*(\iota)$ induces an isomorphism

$$H^*(\text{Hom}_{F[\pi]}(I^{\bar{n}}C_*(\tilde{X}; F), F[\pi])) \rightarrow H_{n-*}(I^{\bar{n}}C_*(\tilde{X}; F)).$$

Consider the diagram

$$\begin{array}{ccc} H^*(\text{Hom}_{F[\pi]}(I^{\bar{n}}C_*(\tilde{X}; F), F[\pi])) & \xrightarrow{\backslash \tilde{d}(\Gamma_X)} & H_{n-*}(I^{\bar{m}}C_*(\tilde{X}; F)) \\ & \searrow \psi_*(\iota) & \downarrow \\ & & H_{n-*}(I^{\bar{n}}C_*(\tilde{X}; F)). \end{array}$$

The map \tilde{d} was defined in Section 3 and \bar{m} denotes $D\bar{n}$ (the lower middle perversity). The vertical arrow is an isomorphism because X is an F -Witt space (see [21, Section 5.6.1]) and the horizontal arrow is an isomorphism by Theorem 4.1, so it suffices to show that the diagram commutes.

For this it suffices to show that the lower horizontal arrow in the following commutative diagram takes $\tilde{d}(\Gamma_X)$ to $\psi_*(\iota)$.

$$\begin{array}{ccc}
H_*(F \otimes_{F[\pi]} I^{\bar{0}} C_*(\tilde{X}; F)) & \xrightarrow{=} & H_*(F \otimes_{F[\pi]} I^{\bar{0}} C_*(\tilde{X}; F)) \\
\downarrow 1 \otimes d & & \downarrow 1 \otimes d \\
H_*(F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{m}}} C_*(\tilde{X} \times \tilde{X}; F)) & \longrightarrow & H_*(F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{n}}} C_*(\tilde{X} \times \tilde{X}; F)) \\
\cong \uparrow 1 \otimes \times & & \cong \uparrow 1 \otimes \times \\
H_*(F \otimes_{F[\pi]} (I^{\bar{n}} C_*(\tilde{X}; F) \otimes_F I^{\bar{m}} C_*(\tilde{X}; F))) & \longrightarrow & H_*(F \otimes_{F[\pi]} (I^{\bar{n}} C_*(\tilde{X}; F) \otimes_F I^{\bar{n}} C_*(\tilde{X}; F))) \\
\cong \downarrow & & \cong \downarrow \\
H_*((I^{\bar{n}} C_*(\tilde{X}; F))^t \otimes_{F[\pi]} I^{\bar{m}} C_*(\tilde{X}; F)) & \longrightarrow & H_*((I^{\bar{n}} C_*(\tilde{X}; F))^t \otimes_{F[\pi]} I^{\bar{n}} C_*(\tilde{X}; F))
\end{array}$$

The definition of \tilde{d} shows that $\tilde{d}(\Gamma_X)$ is the image of b_X (see Notation 5.9) under the left vertical composite, and the definition of ψ shows that $\psi_*(\iota)$ is the image of b_X under the right vertical composite, which completes the proof. \square

5.4 Properties of the symmetric signature for F -Witt spaces

We begin by showing (Proposition 5.13) that σ_{Witt}^* is consistent with the usual symmetric signature σ^* when X is a manifold and (Proposition 5.14) that σ_{Witt}^* is consistent with the Witt class w of X , as defined in [33, Section I.4] and [14, Section 4.1]. We then show that σ_{Witt}^* is additive with respect to disjoint union (Proposition 5.15) and multiplicative with respect to Cartesian product (Theorem 5.16). Next we show that σ_{Witt}^* is invariant under oriented PL homeomorphism and oriented stratified homotopy equivalence (Theorem 5.17) and Witt bordism (Theorem 5.19). Finally, we note that σ_{Witt}^* agrees rationally with the signature index class constructed in [1, Theorem 1.1]; it would be interesting to know whether they also agree integrally.

Proposition 5.13. *If X is a compact oriented manifold and $f : X \rightarrow B\pi$ is a map then $\sigma_{\text{Witt}}^*(f)$ is equal to the usual symmetric signature $\sigma^*(f)$.*

Proof. This is immediate from Corollary 5.8. \square

For our next result, we recall that there is a map $L^n(F[\pi]) \rightarrow L^n(F)$ which takes the class of

$$(C, W \xrightarrow{\phi} C^t \otimes_{F[\pi]} C)$$

to the class of¹

$$(C/\pi, W \xrightarrow{\phi} C^t \otimes_{F[\pi]} C \rightarrow C/\pi \otimes_F C/\pi).$$

Moreover, if $n \equiv 0 \pmod{4}$, or if $\text{char}(F) = 2$ and $n \equiv 0 \pmod{2}$, we can construct a map $L^n(F) \rightarrow W(F)$ (where $W(F)$ is the Witt group) as follows: a symmetric Poincaré complex

¹Here C/π is a convenient shorthand for $F \otimes_{F[\pi]} C$, where F is given the trivial π action.

(D, ψ) over F determines an inner product

$$H_{n/2}(\mathrm{Hom}_F(D, F)) \otimes_F H_{n/2}(\mathrm{Hom}_F(D, F)) \rightarrow F$$

which takes $\alpha \otimes \beta$ to $(\alpha \otimes \beta)(\psi_*(\iota))$, and the proof of [8, Proposition VIII.9.6] shows that the element of $W(F)$ represented by this inner product depends only on the bordism class of (D, ψ) .

Proposition 5.14. *Let X be an F -Witt space of dimension n , where $n \equiv 0 \pmod{4}$ or $\mathrm{char}(F) = 2$ and $n \equiv 0 \pmod{2}$. The composite*

$$L^n(F[\pi]) \rightarrow L^n(F) \rightarrow W(F)$$

takes $\sigma_{\mathrm{Witt}}^(f : X \rightarrow B\pi)$ to the Witt class $w(X)$ (that is, the class of the intersection form on $I^{\bar{n}}H_{n/2}(X; F)$).*

Proof. Let $(I^{\bar{n}}C_*(\tilde{X}; F), \psi)$ be a representative for $\sigma_{\mathrm{Witt}}^*(f : X \rightarrow B\pi)$, where ψ satisfies the condition of Proposition 5.10. The image of $\sigma_{\mathrm{Witt}}^*(f : X \rightarrow B\pi)$ in $L^n(F)$ is represented by $(I^{\bar{n}}C_*(\tilde{X}; F)/\pi, \omega)$, where ω is the composite

$$W \xrightarrow{\psi} I^{\bar{n}}C_*(\tilde{X}; F)^t \otimes_{F[\pi]} I^{\bar{n}}C_*(\tilde{X}; F) \rightarrow I^{\bar{n}}C_*(\tilde{X}; F)/\pi \otimes_F I^{\bar{n}}C_*(\tilde{X}; F)/\pi.$$

Let ω' be the composite

$$W \xrightarrow{\omega} I^{\bar{n}}C_*(\tilde{X}; F)/\pi \otimes_F I^{\bar{n}}C_*(\tilde{X}; F)/\pi \rightarrow I^{\bar{n}}C_*(X; F) \otimes_F I^{\bar{n}}C_*(X; F).$$

The map $I^{\bar{n}}C_*(\tilde{X}; F)/\pi \rightarrow I^{\bar{n}}C_*(X; F)$ is a chain homotopy equivalence by Proposition 6.1.3, and so $(I^{\bar{n}}C_*(X; F), \omega')$ is bordant to $(I^{\bar{n}}C_*(\tilde{X}; F)/\pi, \omega)$ by [29, Proposition 1.13 and Definition 1.6(ii)]. It is straightforward to check that $\omega'_*(\iota)$ is the element $\bar{d}(\Gamma_X)$, where \bar{d} is the algebraic diagonal defined in [18, Section 4.1]. Thus the image of $\sigma_{\mathrm{Witt}}^*(f : X \rightarrow B\pi)$ in $W(F)$ is represented by the inner product

$$I_{\bar{n}}H^{n/2}(X; F) \otimes_F I_{\bar{n}}H^{n/2}(X; F) \rightarrow F$$

which takes $\alpha \otimes \beta$ to $(\alpha \otimes \beta)\bar{d}(\Gamma_X) = (\alpha \cup \beta)(\Gamma_X)$. By [18, Proposition 4.19] and [19, Corollary 5.3] we see that the Poincaré duality isomorphism $I_{\bar{n}}H^{n/2}(X; F) \rightarrow I^{\bar{m}}H_{n/2}(X; F)$ takes this inner product to the intersection form. \square

Proposition 5.15. *If X and Y are F -Witt spaces of the same dimension and $f : X \rightarrow B\pi$, $g : Y \rightarrow B\pi$ are maps, then*

$$\sigma_{\mathrm{Witt}}^*(f \amalg g : X \amalg Y \rightarrow B\pi) = \sigma_{\mathrm{Witt}}^*(f) + \sigma_{\mathrm{Witt}}^*(g).$$

Proof. The proof is a straightforward diagram chase using Definitions 5.3 and 5.11. \square

For our next result we need the multiplication map

$$L^m(F[\pi]) \otimes L^n(F[\rho]) \rightarrow L^{m+n}(F[\pi \times \rho])$$

(see [30, Proposition 8.1]).

Proposition 5.16. *If X and Y are F -Witt spaces and $f : X \rightarrow B\pi$, $g : Y \rightarrow B\rho$ are maps, then*

$$\sigma_{\text{Witt}}^*(f \times g) = \sigma_{\text{Witt}}^*(f) \cdot \sigma_{\text{Witt}}^*(g).$$

Proof. Let $(I^{\bar{n}}C_*(\tilde{X}; F), \psi_X)$ and $(I^{\bar{n}}C_*(\tilde{Y}; F), \psi_Y)$ be representatives for $\sigma_{\text{Witt}}^*(f)$ and $\sigma_{\text{Witt}}^*(g)$. Recall the map

$$\Delta : W \rightarrow W \otimes W$$

defined on page 174 of [30]. The product $\sigma_{\text{Witt}}^*(f) \cdot \sigma_{\text{Witt}}^*(g)$ is (by definition) represented by

$$(I^{\bar{n}}C_*(\tilde{X}; F) \otimes_F I^{\bar{n}}C_*(\tilde{Y}; F), \omega),$$

where ω is the composite

$$\begin{aligned} W &\xrightarrow{\Delta} W \otimes W \xrightarrow{\phi \otimes \psi} (I^{\bar{n}}C_*(\tilde{X}; F)^t \otimes_{F[\pi]} I^{\bar{n}}C_*(\tilde{X}; F)) \otimes (I^{\bar{n}}C_*(\tilde{Y}; F)^t \otimes_{F[\rho]} I^{\bar{n}}C_*(\tilde{Y}; F)) \\ &\cong (I^{\bar{n}}C_*(\tilde{X}; F) \otimes I^{\bar{n}}C_*(\tilde{Y}; F))^t \otimes_{F[\pi \times \rho]} (I^{\bar{n}}C_*(\tilde{X}; F) \otimes I^{\bar{n}}C_*(\tilde{Y}; F)). \end{aligned}$$

By [13, page 382], the cross product induces a map

$$I^{\bar{n}}C_*(\tilde{X}; F) \otimes_F I^{\bar{n}}C_*(\tilde{Y}; F) \rightarrow I^{\bar{n}}C_*(\tilde{X} \times \tilde{Y}; F)$$

and by [29, Proposition 1.13] it suffices to show that this map is a homotopy equivalence ([29, Definition 1.6(ii)]) from $(I^{\bar{n}}C_*(\tilde{X}; F) \otimes_F I^{\bar{n}}C_*(\tilde{Y}; F), \omega)$ to a representative for $\sigma_{\text{Witt}}^*(f \times g)$. For this in turn it suffices to show that the composite

$$\begin{aligned} W &\xrightarrow{\omega} (I^{\bar{n}}C_*(\tilde{X}; F) \otimes_F I^{\bar{n}}C_*(\tilde{Y}; F))^t \otimes_{F[\pi \times \rho]} (I^{\bar{n}}C_*(\tilde{X}; F) \otimes_F I^{\bar{n}}C_*(\tilde{Y}; F)) \\ &\rightarrow I^{\bar{n}}C_*(\tilde{X} \times \tilde{Y}; F)^t \otimes_{F[\pi \times \rho]} I^{\bar{n}}C_*(\tilde{X} \times \tilde{Y}; F) \end{aligned}$$

represents the homology class $c_{f \times g}$ defined in Section 5.3, and this can be verified by a straightforward diagram chase. \square

For part 1 of the following theorem, we use the definition of oriented homeomorphism of stratified pseudomanifolds [18, Definition 5.21]. For part 2, we use the definition of stratified homotopy equivalence given in [18, Appendix A], and we also use the fact [18, Corollary 5.16] that if $g : Y \rightarrow X$ is a stratified homotopy equivalence between compact stratified pseudomanifolds then an orientation of X determines an orientation of Y .

Theorem 5.17. *Let X and Y be F -Witt spaces with maps $f : X \rightarrow B\pi$ and $g : Y \rightarrow X$.*

1. *If g is an oriented PL homeomorphism then $\sigma_{\text{Witt}}^*(f \circ g) = \sigma_{\text{Witt}}^*(f)$.*
2. *Suppose that g is a stratified homotopy equivalence, and give Y the orientation determined by that of X . Then $\sigma_{\text{Witt}}^*(f \circ g) = \sigma_{\text{Witt}}^*(f)$.*

Proof. For part 1, let $|X|$ denote the underlying PL space of X and let X' be the stratification of $|X|$ determined by g . It suffices to show that

$$\sigma_{\text{Witt}}^*(f : X' \rightarrow B\pi) = \sigma_{\text{Witt}}^*(f : X \rightarrow B\pi).$$

Choose a triangulation T of $|X|$ which is compatible with both stratifications. By the proof of [5, Proposition I.1.4], we can define a third stratified pseudomanifold structure X'' for $|X|$ by letting $(X'')^i$ be the i -skeleton of T for $i \leq n-2$ and $|X|$ for $i = n, n-1$. This is a refinement of both X and X' , so we have inclusions

$$i_1 : I^{\bar{n}}C_*(\tilde{X}''; F) \hookrightarrow I^{\bar{n}}C_*(\tilde{X}; F)$$

and

$$i_2 : I^{\bar{n}}C_*(\tilde{X}''; F) \hookrightarrow I^{\bar{n}}C_*(\tilde{X}'; F)$$

which are quasi-isomorphisms (by [20, page 148]) and hence chain homotopy equivalences over $F[\pi]$ (by Proposition 6.4.1 and [22, Exercise IV.4.2]). Choose a representative $(I^{\bar{n}}C_*(\tilde{X}''; F), \psi)$ for $\sigma_{\text{Witt}}^*(f : X'' \rightarrow B\pi)$. By [18, Corollary 5.21], $(I^{\bar{n}}C_*(\tilde{X}; F), (i_1 \otimes i_1)\psi)$ and $(I^{\bar{n}}C_*(\tilde{X}'; F), (i_2 \otimes i_2)\psi)$ are representatives for $\sigma_{\text{Witt}}^*(f : X \rightarrow B\pi)$ and $\sigma_{\text{Witt}}^*(f : X' \rightarrow B\pi)$; moreover, i_1 and i_2 are homotopy equivalences of symmetric Poincaré complexes, so by [29, Proposition 1.13] we have

$$\sigma_{\text{Witt}}^*(f : X \rightarrow B\pi) = \sigma_{\text{Witt}}^*(f : X'' \rightarrow B\pi) = \sigma_{\text{Witt}}^*(f : X' \rightarrow B\pi).$$

For part 2, let $(I^{\bar{n}}C_*(\tilde{Y}; F), \psi)$ be a representative for $\sigma_{\text{Witt}}^*(f \circ g)$. Then $(I^{\bar{n}}C_*(\tilde{X}; F), (g_* \otimes g_*)\psi)$ is a representative for $\sigma_{\text{Witt}}^*(f)$ and (by [11, Proposition 2.1]) g_* is a homotopy equivalence of symmetric Poincaré complexes. \square

Remark 5.18. It seems likely that part 1 is true for all homeomorphisms, not just PL homeomorphisms. The natural way to try to prove this would be to use results of [23]. The obstacle is that the intrinsic coarsest stratification X^* defined by King is a CS space but not a stratified pseudomanifold, and we have not been able to extend the Künneth theorem of [13] to CS spaces.

For our next result, we recall the definition of F -Witt space with boundary ([14, Section 4.1]) and the fact that the orientation of an F -Witt space with boundary induces an orientation of the boundary ([18, Section 7.2]).

Theorem 5.19. *Let X be an F -Witt space with boundary, and let $f : X \rightarrow B\pi$ be a map. Let Y be the boundary of X with the induced orientation. Then $\sigma_{\text{Witt}}^*(f|_Y) = 0$.*

Proof. The idea of the proof is to use the method of Section 5.3 to construct a symmetric Poincaré pair ([29, Definition 1.7]) from the pair (X, Y) .

It's convenient to introduce some notation: given chain complexes C and D and a chain map $f : C \rightarrow D$, we write $H_*(D, C)$ for the homology of the mapping cone Cf (this agrees with the usual meaning of $H_*(D, C)$ when f is a monomorphism). An element of $H_*(D, C)$ is represented by a pair (d, c) with $\partial c = 0$ and $\partial d = -f(c)$.

By Proposition 6.1.3 and the five lemma, the map

$$H_n(F \otimes_{F[\pi]} I^{\bar{0}} C_*(\tilde{X}), F \otimes_{F[\pi]} I^{\bar{0}} C_*(\tilde{Y})) \rightarrow I^{\bar{0}} H_n(X, Y; F)$$

is an isomorphism. Let b map to the fundamental class of X ([18, Section 7.2]) under this isomorphism, and let (η, θ) be a cycle representing b . Let

$$c \in H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, F \otimes_{F[\pi]} I^{\bar{0}} C_*(\tilde{X} \times \tilde{X})), \text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, F \otimes_{F[\pi]} I^{\bar{0}} C_*(\tilde{Y} \times \tilde{Y})))$$

be the class represented by the pair of maps

$$W \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\eta} F \otimes_{F[\pi]} I^{\bar{0}} C_*(\tilde{X}) \xrightarrow{1 \otimes d} F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{n}}} C_*(\tilde{X} \times \tilde{X})$$

and

$$W \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\theta} F \otimes_{F[\pi]} I^{\bar{0}} C_*(\tilde{Y}) \xrightarrow{1 \otimes d} F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{n}}} C_*(\tilde{Y} \times \tilde{Y}).$$

As in Section 5.3, there is an isomorphism

$$\begin{aligned} \Upsilon : H_*(\text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, (I^{\bar{n}} C_*(\tilde{X}; F))^t \otimes_{F[\pi]} I^{\bar{n}} C_*(\tilde{X}; F)), \\ \text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, (I^{\bar{n}} C_*(\tilde{Y}; F))^t \otimes_{F[\pi]} I^{\bar{n}} C_*(\tilde{Y}; F))) \\ \xrightarrow{\cong} H_*(\text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{n}}} C_*(\tilde{X} \times \tilde{X}; F)), \\ \text{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(W, F \otimes_{F[\pi]} I^{Q_{\bar{n}, \bar{n}}} C_*(\tilde{Y} \times \tilde{Y}; F))). \end{aligned}$$

Let

$$\psi : W \rightarrow (I^{\bar{n}} C_*(\tilde{X}; F))^t \otimes_{F[\pi]} I^{\bar{n}} C_*(\tilde{X}; F)$$

be a $\mathbb{Z}/2$ -equivariant chain map that represents $\Upsilon^{-1}(c)$. The proof of Proposition 5.10 adapts (using Theorem 4.5) to show that

$$((I^{\bar{n}} C_*(\tilde{X}; F), \psi), (I^{\bar{n}} C_*(\tilde{Y}; F), \partial\psi))$$

is a symmetric Poincaré pair, and since $(I^{\bar{n}} C_*(\tilde{Y}; F), \partial\psi)$ is a representative for $\sigma_{\text{Witt}}^*(f|_Y)$ (by [18, Proposition 7.9]), we see that $\sigma_{\text{Witt}}^*(f|_Y) = 0$. \square

Finally, recall from [1, Theorem 1.1] the signature index class

$$\text{Ind}(\tilde{\partial}_{\text{sign}}) \in K_*(C_f^* \pi)$$

associated to a smoothly stratified \mathbb{Q} -Witt space X with a map $f : X \rightarrow B\pi$. Also recall from [2, Section 7.2] the map

$$\nu\beta_{\mathbb{Q}} : L^*(\mathbb{Q}\pi) \rightarrow K_*(C_f^* \pi).$$

Proposition 5.20. *$\text{Ind}(\tilde{\partial}_{\text{sign}})$ and $\nu\beta_{\mathbb{Q}}(\sigma_{\text{Witt}}^*(f))$ are equal in $K_*(C_f^* \pi) \otimes \mathbb{Q}$.*

Proof. This follows from Proposition 5.13, Proposition 5.15, Theorem 5.19, and the proof of [2, Proposition 7.1]. \square

6 Technical facts about intersection chains

In this section, we prove some results that were needed in previous sections and in [18].

Throughout this section we fix an n -dimensional ∂ -stratified pseudomanifold X and a regular cover $p : \tilde{X} \rightarrow X$. We write π for the group of covering translations. For any subset S of X we write \tilde{S} for $p^{-1}(S)$.

Recall that an open set U in X is called *evenly covered* if the restriction of the covering map p to U is trivial.

We also fix a perversity \bar{p} .

6.1 A colimit formula for intersection chains

Let \mathcal{U} be a covering of X by open sets. Let \mathcal{C} be the category of all finite intersections of sets in \mathcal{U} , with inclusions as the morphisms. Let A be an open subset of X . Fix a ring R and an R -module M .

Our main result in this subsection is

Proposition 6.1. 1. *The canonical map*

$$\varinjlim_{V \in \mathcal{C}} I^{\bar{p}}C_*(V, V \cap A; M) \rightarrow I^{\bar{p}}C_*(X, A; M)$$

is a chain homotopy equivalence.

2. *The canonical map*

$$\varinjlim_{V \in \mathcal{C}} I^{\bar{p}}C_*(\tilde{V}, \tilde{V} \cap \tilde{A}; M) \rightarrow I^{\bar{p}}C_*(\tilde{X}, \tilde{A}; M)$$

is a chain homotopy equivalence over $R[\pi]$.

3. *The projection*

$$R \otimes_{R[\pi]} I^{\bar{p}}C_*(\tilde{X}, \tilde{A}; M) \rightarrow I^{\bar{p}}C_*(X, A; M)$$

is a chain homotopy equivalence over R .

Remark 6.2. It's possible that the projection in part 3 is actually an isomorphism, as it is for ordinary singular chains.

For the proof of Proposition 6.1 we need a preliminary result which may be of interest in its own right. Let $I_{\mathcal{U}}^{\bar{p}}C_*(X, A; M)$ denote the submodule

$$\sum_{U \in \mathcal{U}} I^{\bar{p}}C_*(U, U \cap A; M)$$

of $I^{\bar{p}}C_*(X, A; M)$.

Proposition 6.3. *The canonical map*

$$\varinjlim_{V \in \mathcal{C}} I^{\bar{p}}C_*(V, V \cap A; M) \rightarrow I_{\mathcal{U}}^{\bar{p}}C_*(X, A; M)$$

is an isomorphism.

The proof of Proposition 6.3 will be given in Subsection 6.4.

Proof of Proposition 6.1. Proposition 2.9 of [12] states that the inclusion

$$I_{\mathcal{U}}^{\bar{P}}C_*(X; M) \hookrightarrow I^{\bar{P}}C_*(X; M)$$

is a chain homotopy equivalence. The same proof shows that the inclusion

$$I_{\mathcal{U}}^{\bar{P}}C_*(X, A; M) \hookrightarrow I^{\bar{P}}C_*(X, A; M)$$

is a chain homotopy equivalence, and part 1 follows from this and Proposition 6.3.

A minor modification of the proof in [12] shows that the inclusion

$$I_{\mathcal{U}}^{\bar{P}}C_*(\tilde{X}, \tilde{A}; M) \hookrightarrow I^{\bar{P}}C_*(\tilde{X}, \tilde{A}; M)$$

is a chain homotopy equivalence over $R[\pi]$, and part 2 follows from this and Proposition 6.3 (applied to the pair (\tilde{X}, \tilde{A})).

For part 3 we assume that the open sets in \mathcal{U} are evenly covered. With this assumption the projection

$$\begin{aligned} R \otimes_{R[\pi]} \left(\varinjlim_{V \in \mathcal{C}} I^{\bar{P}}C_*(\tilde{V}, \tilde{V} \cap \tilde{A}; M) \right) &= \varinjlim_{V \in \mathcal{C}} R \otimes_{R[\pi]} I^{\bar{P}}C_*(\tilde{V}, \tilde{V} \cap \tilde{A}; M) \\ &\rightarrow \varinjlim_{V \in \mathcal{C}} I^{\bar{P}}C_*(V, V \cap A; M) \end{aligned}$$

is an isomorphism. Now consider the diagram

$$\begin{array}{ccc} R \otimes_{R[\pi]} \left(\varinjlim_{V \in \mathcal{C}} I^{\bar{P}}C_*(\tilde{V}, \tilde{V} \cap \tilde{A}; M) \right) & \xrightarrow{\cong} & \varinjlim_{V \in \mathcal{C}} I^{\bar{P}}C_*(V, V \cap A; M) \\ \downarrow & & \downarrow \\ R \otimes_{R[\pi]} I^{\bar{P}}C_*(\tilde{X}, \tilde{A}; M) & \longrightarrow & I^{\bar{P}}C_*(X, A; M) \end{array}$$

The right vertical arrow is a chain homotopy equivalence over R by part 1. The left vertical arrow is a chain homotopy equivalence over R by part 2. Hence the lower horizontal arrow is a chain homotopy equivalence over R as required. \square

6.2 Freeness and flatness

In this section we prove two results. Let A be an open subset of X and let F be a field.

First we have

Proposition 6.4. *1. If X has a finite covering by evenly covered open sets (in particular, if X is compact) then $I^{\bar{P}}C_*(\tilde{X}, \tilde{A}; F)$ is chain homotopy equivalent over $F[\pi]$ to a nonnegatively-graded chain complex of free $F[\pi]$ -modules.*

2. For all X , $I^{\bar{P}}C_(\tilde{X}, \tilde{A}; F)$ is chain homotopy equivalent over $F[\pi]$ to a nonnegatively-graded chain complex of flat $F[\pi]$ -modules.*

For the second result we let π act by the diagonal action on $I^{\bar{p}}C_*(\tilde{X}, \tilde{A}; F) \otimes_F I^{\bar{q}}C_*(\tilde{X}, \tilde{A}; F)$ and on $I^{Q_{\bar{p}, \bar{q}}}C_*(\tilde{X} \times \tilde{X}, \tilde{A} \times \tilde{X} \cup \tilde{X} \times \tilde{A}; F)$.

Proposition 6.5. *The cross product*

$$I^{\bar{p}}C_*(\tilde{X}, \tilde{A}; F) \otimes_F I^{\bar{q}}C_*(\tilde{X}, \tilde{A}; F) \rightarrow I^{Q_{\bar{p}, \bar{q}}}C_*(\tilde{X} \times \tilde{X}, \tilde{A} \times \tilde{X} \cup \tilde{X} \times \tilde{A}; F)$$

induces a quasi-isomorphism

$$F \otimes_{F[\pi]} (I^{\bar{p}}C_*(\tilde{X}, \tilde{A}; F) \otimes_F I^{\bar{q}}C_*(\tilde{X}, \tilde{A}; F)) \rightarrow F \otimes_{F[\pi]} I^{Q_{\bar{p}, \bar{q}}}C_*(\tilde{X} \times \tilde{X}, \tilde{A} \times \tilde{X} \cup \tilde{X} \times \tilde{A}; F).$$

For both results we will give the proofs when $A = \emptyset$; the same proofs work for the general cases.

For the proof of Proposition 6.4 we need a lemma.

Lemma 6.6. *Let \mathcal{V} be a finite collection of evenly covered open sets in X . Let \mathcal{D} be the category of intersections of sets of \mathcal{V} , with inclusions as the morphisms. Then*

$$\varinjlim_{V \in \mathcal{D}} I^{\bar{p}}C_*(\tilde{V}; F)$$

is free over $F[\pi]$.

Proof. For each W in \mathcal{D} let $A(W)$ be the image of the map

$$\varinjlim I^{\bar{p}}C_*(\tilde{V}; F) \rightarrow I^{\bar{p}}C_*(\tilde{W}; F),$$

where the colimit is taken over $V \in \mathcal{D}$ with $V \subsetneq W$, and let $B(W)$ be the cokernel of $A(W) \rightarrow I^{\bar{p}}C_*(\tilde{W}; F)$. $B(W)$ is free over $F[\pi]$ because W is evenly covered and F is a field. It follows that the short exact sequence

$$0 \rightarrow A(W) \rightarrow I^{\bar{p}}C_*(\tilde{W}; F) \rightarrow B(W) \rightarrow 0$$

is split, and from this it follows that $\varinjlim_{V \in \mathcal{D}} I^{\bar{p}}C_*(\tilde{V}; F)$ is isomorphic to $\oplus_{W \in \mathcal{D}} B(W)$ (observe inductively that $A(W)$ is the direct sum $\oplus_{V \subsetneq W} B(V)$ and that the colimit identifies the various copies of $B(V)$ in the obvious way). \square

Proof of Proposition 6.4. Part 1 is immediate from Lemma 6.6 and Proposition 6.1.2.

For part 2, let \mathcal{U} be a collection of evenly covered open sets whose union is X , and let \mathcal{C} be the category of finite intersections of sets in \mathcal{U} . For each finite subset \mathcal{V} of \mathcal{U} let $\mathcal{D}(\mathcal{V})$ be the category of intersections of sets in \mathcal{V} . Then

$$\varinjlim_{V \in \mathcal{C}} I^{\bar{p}}C_*(\tilde{V}; F) = \varinjlim_{\mathcal{V}} \varinjlim_{V \in \mathcal{D}(\mathcal{V})} I^{\bar{p}}C_*(\tilde{V}; F)$$

and the result follows from Proposition 6.1.2, Lemma 6.6, and the fact that a directed colimit of flat modules is flat. \square

Proof of Proposition 6.5. Let C and D denote $I^{\bar{p}}C_*(\tilde{X}; F) \otimes_F I^{\bar{q}}C_*(\tilde{X}; F)$ and $I^{\mathcal{Q}_{\bar{p}, \bar{q}}}C_*(\tilde{X} \times \tilde{X}; F)$ respectively. Let R denote $F[\pi \times \pi]$, which is isomorphic to $F[\pi] \otimes F[\pi]$. By Proposition 6.4.2, we have chain homotopy equivalences $C \rightarrow C'$ and $D \rightarrow D'$ over R (and hence over $F[\pi]$), where C' and D' are nonnegatively-graded and flat over R . But R is flat (in fact free) over $F[\pi]$, and hence C' and D' are flat over $F[\pi]$ (because the functor $C' \otimes_{F[\pi]} -$ is naturally isomorphic to $C' \otimes_R R \otimes_{F[\pi]} -$, and similarly for D'). Now the map

$$C \xrightarrow{\sim} D$$

is a quasi-isomorphism by the Künneth theorem of [13], and hence the composite

$$C' \rightarrow C \rightarrow D \rightarrow D'$$

induces a quasi-isomorphism

$$F \otimes_{F[\pi]} C' \rightarrow F \otimes_{F[\pi]} D'$$

by [34, Theorem 5.6.4]. The maps $F \otimes_{F[\pi]} C' \rightarrow F \otimes_{F[\pi]} C$ and $F \otimes_{F[\pi]} D \rightarrow F \otimes_{F[\pi]} D'$ are quasi-isomorphisms because $F \otimes_{F[\pi]}$ preserves chain homotopy equivalences over $F[\pi]$, so we conclude that $F \otimes_{F[\pi]} C \rightarrow F \otimes_{F[\pi]} D$ is a quasi-isomorphism as required. \square

6.3 Proof of Proposition 5.12

Recall (for example from [22, Exercise IV.4.2]) that if two bounded-below chain complexes are free over $F[\pi]$ and quasi-isomorphic over $F[\pi]$ then they are chain homotopy equivalent over $F[\pi]$. Combining this with Proposition 6.4.1, it suffices to show that $I^{\bar{p}}C_*(\tilde{X}; F)$ is quasi-isomorphic over $F[\pi]$ to a finite $F[\pi]$ chain complex.

This in turn is immediate from the following lemma. Let X' denote $X - \partial X$.

Lemma 6.7. 1. *The map*

$$I^{\bar{p}}H_*(\tilde{X}'; F) \rightarrow I^{\bar{p}}H_*(\tilde{X}; F)$$

induced by the inclusion is an isomorphism.

2. *$I^{\bar{p}}C_*(\tilde{X}'; F)$ is quasi-isomorphic over $F[\pi]$ to a finite $F[\pi]$ chain complex.*

Remark 6.8. The reason that X' plays a special role is that we will need to use the relation between intersection homology and the Deligne sheaf, and this relation is not known for ∂ -stratified pseudomanifolds with nonempty boundary.

Before continuing we need to recall some definitions. Let K be a simplicial complex. A subcomplex L of K is *full* if every simplex whose vertices are in L is in L . Let s be a simplex of K . The *closed star* of s is the union of all the simplices containing it; this will be denoted $\overline{\text{St}}(s)$. The *open star* of s is the interior of $\overline{\text{St}}(s)$; this will be denoted $\text{St}(s)$.

Fix a triangulation of X with the property that each skeleton of X is a full subcomplex.

For the proof of Lemma 6.7 we need two other lemmas, whose proofs we defer for a moment.

Lemma 6.9. *Let s be a simplex of X which is contained in X' . Then $I^{\bar{p}}C_*(\widetilde{\text{St}(s)}; F)$ is homotopy finite over $F[\pi]$.*

Lemma 6.10. *The homotopy pushout (double mapping cylinder) of homotopy finite chain complexes over $F[\pi]$ is quasi-isomorphic over $F[\pi]$ to a finite $F[\pi]$ chain complex.*

Proof of Lemma 6.7. Part 1. By the definition of ∂ -stratified pseudomanifold ([18, Definition 7.1]) ∂X has an open collar neighborhood in X . This implies that the inclusion $\widetilde{X}' \rightarrow \tilde{X}$ is a stratified homotopy equivalence, and the result follows from [18, Appendix A].

Part 2. First observe that X' is the union of the open stars of the vertices of X that are contained in X' and that there are finitely many such vertices (because X is compact). We will also use the fact that the intersection of the open stars of finitely many vertices, if it is nonempty, is the open star of the simplex determined by these vertices.

We will prove by induction on k that if U_1, \dots, U_k are open stars of simplices contained in X' and U is $U_1 \cup \dots \cup U_k$ then $I^{\bar{p}}C_*(\tilde{U}; F)$ is quasi-isomorphic over $F[\pi]$ to a finite $F[\pi]$ chain complex. Let $V = U_1 \cup \dots \cup U_{k-1}$ and let $W = V \cap U_k$. Let C be the pushout of the diagram

$$\begin{array}{ccc} I^{\bar{p}}C_*(\tilde{W}; F) & \longrightarrow & I^{\bar{p}}C_*(\tilde{U}_k; F) \\ \downarrow & & \\ I^{\bar{p}}C_*(\tilde{V}; F) & & \end{array} \quad (*)$$

and let D be its homotopy pushout. $I^{\bar{p}}C_*(\tilde{U}; F)$ is chain homotopy equivalent to C by Proposition 6.1.2. The three chain complexes in diagram (*) are homotopy finite over $F[\pi]$ (this follows from the inductive hypothesis, Proposition 6.4.1, and Lemma 6.9) so by Lemma 6.10 D is quasi-isomorphic over $F[\pi]$ to a finite $F[\pi]$ chain complex. To conclude the proof we show that the quotient map $D \rightarrow C$ is a quasi-isomorphism. Diagram (*) gives a Mayer-Vietoris sequence

$$\dots \rightarrow I^{\bar{p}}H_i(\tilde{W}; F) \rightarrow I^{\bar{p}}H_i(\tilde{U}_k; F) \oplus I^{\bar{p}}H_i(\tilde{V}; F) \rightarrow H_i(D) \rightarrow I^{\bar{p}}H_{i-1}(\tilde{W}; F) \rightarrow \dots$$

There is also a Mayer-Vietoris sequence for C (because the map

$$I^{\bar{p}}C_i(\tilde{W}; F) \rightarrow I^{\bar{p}}C_i(\tilde{U}_k; F) \oplus I^{\bar{p}}C_i(\tilde{V}; F)$$

is a monomorphism) so the five lemma shows that $H_*(D) \rightarrow H_*(C)$ is an isomorphism. \square

For the proof of Lemma 6.9 we need a definition. The *combinatorial link* of s , denoted $\text{Lk}(s)$, is the union of the simplices of $\overline{\text{St}}(s)$ that do not intersect s .

Proof of Lemma 6.9. First recall (for example from [27, Lemma 62.6]) that $\overline{\text{St}}(s)$ is equal to the join $s * \text{Lk}(s)$.

In particular, $\overline{\text{St}}(s)$ is contractible, so the covering map $p : \tilde{X} \rightarrow X$ is trivial over $\text{St}(s)$, and hence

$$I^{\bar{p}}C_*(\widetilde{\text{St}(s)}; F) \cong F[\pi] \otimes I^{\bar{p}}C_*(\text{St}(s); F).$$

Thus it suffices to show that $I^{\bar{p}}C_*(\text{St}(s); F)$ is homotopy finite over F . But (using the fact that F is a field) $I^{\bar{p}}C_*(\text{St}(s); F)$ is chain homotopy equivalent to $I^{\bar{p}}H_*(\text{St}(s); F)$, so it suffices to show that the latter is finitely generated.

Now $s = \hat{s} * \partial s$, where \hat{s} is the barycenter of s , and so $\overline{\text{St}}(s) = \hat{s} * \partial s * \text{Lk}(s)$. This is homeomorphic to the cone on $\partial s * \text{Lk}(s)$, and the homeomorphism takes $\text{St}(s)$ to the open cone

$$([0, 1) \times (\partial s * \text{Lk}(s))) / (0 \times x \sim 0 \times y)$$

which we denote by Q . We give Q the stratification determined by the homeomorphism. Each subspace $(0, 1) \times z$ of Q is taken by the inverse homeomorphism to the interior of a simplex of X , and the interior of each simplex of X is contained in a single stratum, so each subspace $(0, 1) \times z$ is contained in a single stratum of Q . It follows that the subspace

$$([0, 1/2) \times (\partial s * \text{Lk}(s))) / (0 \times x \sim 0 \times y),$$

which we denote by P , is stratified homotopy equivalent to Q (as defined in [18, Appendix A]). Next we recall that $I^{\bar{p}}H_*$ of an open set in X' is the hypercohomology of the Deligne sheaf (for general perversities this is [16, Theorem 3.6]) and that the Deligne sheaf is cohomologically constructible ([16, Proposition 4.1]), which in particular means that it satisfies Wilder's Property (P,Q) ([5, page 69]). In our situation this says that the image of the map $I^{\bar{p}}H_*(P; F) \rightarrow I^{\bar{p}}H_*(Q; F)$ is finitely generated. But this map is an isomorphism by [18, Appendix A], so $I^{\bar{p}}H_*(Q; F)$ is finitely generated as required. \square

Proof of Lemma 6.10. Let

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ f \downarrow & & \\ B & & \end{array} \quad (*)$$

be a diagram of homotopy finite chain complexes over $F[\pi]$ and $F[\pi]$ chain maps. Recall that the homotopy pushout of diagram $(*)$ is defined as follows. Let I denote the cellular chain complex of the unit interval, that is, the F chain complex with two generators a and b in dimension 0, one generator c in dimension 1, and differential $\partial c = b - a$. Let F be the chain complex consisting of F in dimension 0, and let $\alpha, \beta : F \rightarrow I$ be the maps which take 1 to a and b respectively. Define B' by the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{A \otimes \beta} & A \otimes I \\ f \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

and similarly for C' . Then the homotopy pushout of $(*)$, which we will denote by D , is defined by the pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & C' \\ \downarrow & & \downarrow \\ B' & \longrightarrow & D, \end{array}$$

where the upper horizontal and leftmost vertical arrows are induced by $A \otimes \alpha$.

Next let $i : A \rightarrow \bar{A}$, $j : B \rightarrow \bar{B}$, $k : C \rightarrow \bar{C}$ be chain homotopy equivalences over $F[\pi]$ with \bar{A} , \bar{B} , \bar{C} finite $F[\pi]$ chain complexes. Then there are maps $\bar{f} : \bar{A} \rightarrow \bar{B}$ and $\bar{g} : \bar{A} \rightarrow \bar{C}$ making the diagram

$$\begin{array}{ccccc} B & \xleftarrow{f} & A & \xrightarrow{g} & C \\ j \downarrow & & i \downarrow & & k \downarrow \\ \bar{B} & \xleftarrow{\bar{f}} & \bar{A} & \xrightarrow{\bar{g}} & \bar{C} \end{array} \quad (**)$$

commute up to chain homotopy. The homotopy pushout \bar{D} of the second row is given by a pushout diagram

$$\begin{array}{ccc} \bar{A} & \longrightarrow & \bar{C}' \\ \downarrow & & \downarrow \\ \bar{B}' & \longrightarrow & \bar{D}. \end{array}$$

It is easy to check that \bar{D} is a finite $F[\pi]$ chain complex. To compare D with \bar{D} we introduce an “extended” version of D . Define a chain complex $2I$ by the pushout diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & I \\ \beta \downarrow & & \downarrow \delta \\ I & \xrightarrow{\gamma} & 2I. \end{array} \quad (***)$$

Let ζ (resp., η) be the composite $F \xrightarrow{\alpha} I \xrightarrow{\gamma} 2I$ (resp., $F \xrightarrow{\beta} I \xrightarrow{\delta} 2I$). Replacing I by $2I$, α by ζ , and β by η in the construction of D gives a pushout diagram

$$\begin{array}{ccc} A & \longrightarrow & C'' \\ \downarrow & & \downarrow \\ B'' & \longrightarrow & E. \end{array}$$

Next we construct a quasi-isomorphism $E \rightarrow D$. In diagram (***), write $I_1 \subset 2I$ for the image of γ and I_2 for the image of δ . Also let $\epsilon : I \rightarrow F$ be the chain map which takes a and b to 1. Define a map $\theta : 2I \rightarrow I$ by letting θ be γ^{-1} on I_1 and $\beta \circ \epsilon \circ \delta^{-1}$ on I_2 . θ induces maps $B'' \rightarrow B'$ and $C'' \rightarrow C'$ and hence a map $E \rightarrow D$. Applying the five lemma to the Mayer-Vietoris sequences of E and D shows that the map $E \rightarrow D$ is a quasi-isomorphism.

Finally, we construct a quasi-isomorphism $E \rightarrow \bar{D}$. In diagram (**), let $H : A \otimes I \rightarrow \bar{B}$ be the chain homotopy from $\bar{f} \circ i$ to $j \circ f$. Define a map $\kappa : B'' \rightarrow \bar{B}'$ to be $A \otimes \gamma^{-1}$ on $A \otimes I_1$, $H \circ (A \otimes \delta^{-1})$ on $A \otimes I_2$, and j on B . Similarly, define a map $\lambda : C'' \rightarrow \bar{C}$. Then κ and λ give a map $E \rightarrow \bar{D}$, and applying the five lemma to the Mayer-Vietoris sequences of E and \bar{D} shows that this is a quasi-isomorphism. \square

6.4 Proof of Proposition 6.3

We need a lemma, whose proof we defer for a moment.

Lemma 6.11. *Let $U_1, \dots, U_m \in \mathcal{U}$ and let $\xi_i \in I^{\bar{p}}C_*(U_i; M)$ for $1 \leq i \leq m$ with*

$$\sum \xi_i = 0$$

in $I^{\bar{p}}C_(X, A; M)$. Then for $2 \leq i \leq m$ there exist $\eta_i \in I^{\bar{p}}C_*(U_i \cap U_1; M)$ with*

$$\xi_1 + \sum_{i=2}^m \eta_i = 0$$

in $I^{\bar{p}}C_(X, A; M)$.*

Proof of Proposition 6.3. Let K be the kernel of the canonical epimorphism

$$\bigoplus_{U \in \mathcal{U}} I^{\bar{p}}C_*(U, U \cap A; M) \rightarrow I_{\mathcal{U}}^{\bar{p}}C_*(X, A; M)$$

and let L be the kernel of the canonical epimorphism

$$\bigoplus_{U \in \mathcal{U}} I^{\bar{p}}C_*(U, U \cap A; M) \rightarrow \varinjlim_{V \in \mathcal{C}} I^{\bar{p}}C_*(V, V \cap A; M).$$

For a chain $\xi \in I^{\bar{p}}C_*(U; M)$, let $[\xi]$ denote its image in $I^{\bar{p}}C_*(U, U \cap A; M)$. K is generated by tuples

$$([\xi_1], \dots, [\xi_m]) \in \bigoplus_{i=1}^m I^{\bar{p}}C_*(U_i, U_i \cap A; M)$$

with $\sum \xi_i = 0$ in $I^{\bar{p}}C_*(X, A; M)$, as (U_1, \dots, U_m) ranges over all m -tuples in \mathcal{U} . L is generated by pairs

$$([\xi], [-\xi]) \in I^{\bar{p}}C_*(U, U \cap A; M) \oplus I^{\bar{p}}C_*(U', U' \cap A; M)$$

with $\xi \in I^{\bar{p}}C_*(U \cap U'; M)$, as (U, U') ranges over all pairs in \mathcal{U} . It's clear that $L \subset K$ and it suffices to show that each of the generating tuples for K is in L . So let $([\xi_1], \dots, [\xi_m])$ be such a tuple. We assume inductively that all shorter such tuples are in L . Lemma 6.11 gives an equation

$$([\xi_1], \dots, [\xi_m]) = ([-\eta_2], [\eta_2], 0, \dots, 0) + ([-\eta_3], 0, [\eta_3], 0, \dots, 0) + \dots + ([-\eta_m], 0, \dots, 0, [\eta_m]) \\ + (0, [\xi_2 - \eta_2], \dots, [\xi_m - \eta_m]).$$

The last summand on the right is in L by the inductive hypothesis, and the remaining summands are obviously in L . \square

Proof of Lemma 6.11. We begin with the case $A = \emptyset$.

For a chain ξ and a singular simplex σ with the same dimension as ξ , we write

$$c_{\xi}(\sigma)$$

for the coefficient of σ in ξ . We say that σ *belongs* to ξ if $c_{\xi}(\sigma) \neq 0$.

Let U_i and ξ_i , $1 \leq i \leq m$, be as in the lemma. For $2 \leq i \leq m$, let \mathfrak{A}_i be the set of singular simplices which belong to both ξ_i and ξ_1 , and let

$$\theta_i = \sum_{\sigma \in \mathfrak{A}_i} c_{\xi_i}(\sigma) \sigma.$$

The equation $\sum_{i=1}^m \xi_i = 0$ implies

$$\xi_1 + \sum_{i=2}^m \theta_i = 0, \tag{1}$$

which might suggest we could take η_i to be θ_i , but θ_i will not be an intersection chain in general because its boundary can contain non-allowable simplices that cancel out in ξ_i .

For $2 \leq i \leq m$, let \mathfrak{B}_i be the set of singular simplices which belong to ξ_i and intersect U_1 but do not belong to ξ_1 . Let $\mathfrak{B} = \cup_{i=2}^m \mathfrak{B}_i$. The equation $\sum_{i=1}^m \xi_i = 0$ implies

$$\sum_{i=2}^m c_{\xi_i}(\sigma) = 0 \tag{2}$$

for each $\sigma \in \mathfrak{B}$.

The strategy of the rest of the proof is to replace each σ in \mathfrak{B} by a chain $\bar{\sigma}$, in such a way that for $2 \leq i \leq m$

- (I) the support $|\bar{\sigma}|$ is contained in $|\sigma| \cap U_1$, and
- (II) the chain $\theta_i + \sum_{\sigma \in \mathfrak{B}_i} c_{\xi_i}(\sigma) \bar{\sigma}$ is allowable.

We can then let η_i be $\theta_i + \sum_{\sigma \in \mathfrak{B}_i} c_{\xi_i}(\sigma) \bar{\sigma}$; the equation $\xi_1 + \sum_{i=2}^m \eta_i = 0$ will follow from equations (1) and (2).

We will construct the chains $\bar{\sigma}$ by using the subdivision procedure in the proof of [12, Proposition 2.9] (with the ordered cover U_1, X); for the convenience of the reader we give the details.

First we need some notation. Suppose we are given

- a singular simplex $\tau : \Delta^j \rightarrow X$,
- a simplicial complex K which is a subdivision of Δ^j , and
- an ordering of the vertices of K which is a total ordering on the vertices of each simplex.

For each j -dimensional simplex s of K the total ordering of the vertices of s determines an affine isomorphism

$$\mathbf{i}_s : \Delta^j \rightarrow s.$$

Let ϵ_s be 1 if the total ordering of the vertices of s agrees with the orientation inherited from Δ^j and -1 otherwise. Let

$$\mathbf{i}_K = \sum \epsilon_s \mathbf{i}_s, \tag{3}$$

where the sum is taken over all j -dimensional simplices of K . Then \mathbf{i}_K is a singular chain of Δ^j . The chain $\tau_*(\mathbf{i}_K)$ is the subdivision of τ determined by the given data.

Now suppose in addition that τ is allowable. Then [12, Lemma 2.6] says that for every j -dimensional simplex s of K the singular simplex $\tau \circ \mathbf{i}_s$ is allowable. Also, if t is a $(j-1)$ -dimensional simplex of K then a straightforward argument (which is written out on page 1993 of [12]) shows that $\tau \circ \mathbf{i}_t$ is allowable except perhaps when t contains a simplex u which is contained in the $\dim(u)$ -skeleton of Δ^j . We will call a simplex u of K which is contained in the $\dim(u)$ -skeleton of Δ^j *awkward* (with respect to τ).

Let k denote the dimension of the chains ξ_i . For $0 \leq j \leq k$, let \mathfrak{B}^j denote the set of singular simplices of dimension j which are faces of singular simplices in \mathfrak{B} (in particular $\mathfrak{B}^k = \mathfrak{B}$). By induction on j , we will construct for each $\tau \in \mathfrak{B}^j$

- a subdivision K_τ of Δ^j , and
- a partial ordering of the vertices of K_τ which restricts to a total ordering on the vertices of each simplex,

with the following properties.²

- (i) If $|\tau| \subset U_1$ then $K_\tau = \Delta^j$.
- (ii) Under the identification of the l -th face of Δ^j with Δ^{j-1} , the subdivision of the l -th face agrees with $K_{\partial_l \tau}$.
- (iii) If u is an awkward simplex of K_τ which is contained in $\tau^{-1}(U_1)$, then any simplex of K_τ containing u is contained in $\tau^{-1}(U_1)$.

For $j = 0$, $K_\tau = \Delta^0$. Suppose the construction has been accomplished for all dimensions $< j$ and let $\tau \in \mathfrak{B}^j$ with $|\tau|$ not contained in U_1 . The subdivisions associated to the faces of τ give a simplicial complex K_0 which is a subdivision of the boundary of Δ^j . Let Δ' be the cone on K_0 . Then K_0 is a subcomplex of Δ' so we can apply barycentric subdivision holding K_0 fixed (see [27, page 89] for the definition) until Property (iii) is satisfied (see the proof of [27, Lemma 16.3]). We order the vertices at each stage of the subdivision process by letting each new vertex be greater than all the existing vertices adjacent to it.

Now for each $\sigma \in \mathfrak{B}^k$ we let

$$\bar{\sigma} = \sum \epsilon_s \sigma \circ \mathbf{i}_s$$

where the sum is over all simplices s of K_σ that are contained in $\sigma^{-1}(U_1)$. Also, for each $\tau \in \mathfrak{B}^{k-1}$, we let

$$\bar{\tau} = \sum \epsilon_t \tau \circ \mathbf{i}_t$$

where the sum is over all simplices t of K_τ that are contained in $\tau^{-1}(U_1)$.

We need to show that the $\bar{\sigma}$ satisfy Properties (I) and (II) above. Property (I) is clearly satisfied. As a first step toward Property (II), we calculate $\partial \bar{\sigma}$ modulo allowable singular simplices. Fix a $\sigma \in \mathfrak{B}^k$ and let

$$\mathbf{j} = \sum \epsilon_s \mathbf{i}_s,$$

²Property (i) is a slight modification of the procedure in [12].

where the sum is over all simplices of K_σ that are contained in $\sigma^{-1}(U_1)$; then $\bar{\sigma} = \sigma_* \mathbf{j}$ and $\partial \bar{\sigma} = \sigma_*(\partial \mathbf{j})$. Suppose that t is a $(k-1)$ -simplex belonging to $\partial \mathbf{j}$ such that $\sigma \circ \mathbf{i}_t$ is non-allowable. Then t must contain an awkward simplex of K_σ , so Property (iii) implies that the coefficient of \mathbf{i}_t in $\partial \mathbf{j}$ is the same as its coefficient in $\partial \mathbf{i}_{K_\sigma}$ (see equation (3)). If t is not contained in $\partial \Delta^k$ then this coefficient is 0. If t is contained in the l -th face of Δ^k then (identifying this face with Δ^{k-1}) this coefficient is $(-1)^l \epsilon_t$. It follows that

$$\partial \bar{\sigma} \equiv \sum_{\tau \in \mathfrak{B}^{k-1}} c_{\partial \sigma}(\tau) \bar{\tau} \quad (4)$$

modulo allowable singular simplices.

Now we can verify Property (II). Let η_i denote $\theta_i + \sum_{\sigma \in \mathfrak{B}_i} c_{\xi_i}(\sigma) \bar{\sigma}$. All singular simplices that belong to η_i are allowable by [12, Lemma 2.6], so it only remains to check that the singular simplices that belong to $\partial \eta_i$ are allowable. First note that if τ is non-allowable and belongs to $\partial \theta_i$ then τ is an element of \mathfrak{B}^{k-1} (because $\partial \theta_i \subset U_1$ and ξ_i is allowable), and we have $\bar{\tau} = \tau$ by Property (i). This implies that, modulo allowable singular simplices, we have

$$\partial \theta_i \equiv \sum_{\tau \in \mathfrak{B}^{k-1}} c_{\partial \theta_i}(\tau) \bar{\tau}. \quad (5)$$

Combining equations (4) and (5) gives

$$\partial \eta_i \equiv \sum_{\tau \in \mathfrak{B}^{k-1}} \left[c_{\partial \theta_i}(\tau) + \sum_{\sigma \in \mathfrak{B}_i} c_{\xi_i}(\sigma) c_{\partial \sigma}(\tau) \right] \bar{\tau}. \quad (6)$$

If τ is allowable then all singular simplices belonging to $\bar{\tau}$ are allowable, by [12, Lemma 2.6]. If τ is not allowable and $\bar{\tau} \neq 0$ then $|\tau|$ must intersect U_1 , which implies that the expression in brackets in equation (6) is equal to the coefficient of τ in $\partial \xi_i$, which is 0 since ξ_i is allowable. Thus all singular simplices belonging to $\partial \eta_i$ are allowable, as required.

This completes the proof of Lemma 6.11 for the case $A = \emptyset$. For the general case, we are given $\xi_i \in I^{\bar{p}}C_*(U_i; M)$ for $1 \leq i \leq m$ with

$$\sum \xi_i \in I^{\bar{p}}C_*(A; M).$$

Let $U_{m+1} = A$ and $\xi_{m+1} = -\sum_{i=1}^m \xi_i$. Applying the case already proved to the $(m+1)$ -tuple $(\xi_1, \dots, \xi_{m+1})$, we obtain $\eta_i \in I^{\bar{p}}C_*(U_i \cap U_1; M)$ for $2 \leq i \leq m+1$ with

$$\xi_1 + \sum_{i=2}^{m+1} \eta_i = 0,$$

and from this it follows that

$$\xi_1 + \sum_{i=2}^m \eta_i \in I^{\bar{p}}C_*(A; M)$$

as required. \square

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